Some generalizations in steady one-dimensional gas dynamics

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SUMMARY

In a stationary weak sound wave the four gas dynamical quantities, entropy, stagnation enthalpy, mass flow per unit area and impulse per unit area are constant. The six processes in which pairs of these four variables are kept constant are studied for the case of any single phase fluid or mixture of fluids in equilibrium, and it is shown that the remaining variables are stationary at sonic points. Such points are shown to occur once only in each process and are identified as maxima or minima, on the assumption that the fluid is a normal one which expands on heating at constant pressure and for which $(\partial^2 p/\partial v^2)_s$ is positive.

Newton's theory of sound assumed that the fluid temperature was invariant. Across a stationary Newtonian sound wave, the four quantities, temperature, mass flow and impulse per unit area, and a dynamical variant of the Gibbs function are constant. The six processes in which pairs of these four variables are kept constant are studied, and it is shown that the remaining variables are stationary at points where the fluid speed equals Newton's sound velocity. These points are shown to occur once only in each process and are identified as maxima or minima with the further proviso that $(\partial^2 p/\partial v^2)_T$ is positive.

The two sets of processes have one member in common, that usually referred to as the Rayleigh line. The Fanno line and the usual isentropic 'nozzle' process also belong to the first set.

Finally, the variation of stagnation temperature and pressure in some of the processes and in stationary shocks is investigated.

1. INTRODUCTION

Steady, one-dimensional gas dynamics has been analysed by many authors, particularly for the case of a perfect gas. A certain amount of work has also been done for arbitrary, single-phase fluids in equilibrium, with properties limited only by stability and the laws of thermodynamics (Courant & Friedrichs 1948; Kline & Shapiro 1954). It has been shown, for instance, that the speed of sound is a critical velocity for isentropic nozzle flows, and for Rayleigh and Fanno processes, whatever the fluid. This is too striking a fact to be pure coincidence. It is the aim of this paper to construct a symmetrical mathematical treatment of these three and related processes, showing the dominant role of the sound speed.

The importance of studying arbitrary fluid dynamics is increasing since high-speed flows in which real gas behaviour, reaction, dissociation and ionization occur are becoming of practical interest. The theory of the arbitrary fluid in this and other papers is applicable to reactive gas mixtures provided only that at each section of flow investigated there is thermodynamic equilibrium. The speed of sound occurring in the analysis refers to the speed of waves of low enough frequency for chemical equilibrium to prevail. The theory cannot be applied to detonation waves in pre-mixed gases since there is true equilibrium on only one side of these waves. The flow behind and relative to a detonation wave can be sonic according to the Chapman–Jouguet rule, whereas behind a shock wave in a reactive gas mixture, which is in equilibrium on *both* sides of the wave, the flow must be subsonic. This last fact is evident from the work of Kline & Shapiro (1954), provided the gas mixture's properties are such that $(\partial^2 p/\partial v^2)_s$ and $(\partial p/\partial s)_v$ are constant in sign. p, v and s are pressure, specific volume and entropy, respectively.

Gases such as HCl which dissociate without total molal or volumetric change may be treated by perfect gas dynamical methods (Shapiro & Hawthorne 1947) with allowance for variation in effective specific heats.

2. THE GAS DYNAMICAL RELATIONS

Reactive or inert gas mixtures and single phase pure substances in equilibrium are usually characterized by the fact that any two independent properties define their thermodynamic state uniquely. As a generalization of this, the state of each section of flow in a one-dimensional gas dynamical process may be defined by *three* independent properties. One choice would be the velocity u together with any two independent thermodynamic ones. Another selection is F, G and H, in which

$$F = p + \rho u^2, \text{ the impulse per unit area,}$$

$$G = \rho u, \text{ the mass flow rate per unit area, and}$$

$$H = h + \tfrac{1}{2}u^2, \text{ the stagnation enthalpy.}$$

ρ and h denote density and specific enthalpy respectively. For convenience F is chosen equal to $p + \rho u^2$ rather than the more usual quantity

$$(p + \rho u^2) \times (\text{duct area of cross-section}).$$

F/G represents this latter quantity per unit rate of mass flow. The symbol H is chosen rather than h_0 to avoid complicated suffices. Note that $F = p + Gu$.

Specified values of F, G and H usually correspond to *two* possible states, such as occur on the two sides of a stationary shock wave. Across a stationary *weak* shock or acoustic wave, s is constant in addition to F, G and H, a very significant fact.

The familiar reversible, adiabatic 'nozzle' process and Rayleigh and Fanno processes each involve the invariance of two of the properties F, G, H and s. There are three other such processes. All six will be studied simultaneously. With suitable assertions regarding reversibility, adiabatic or other conditions, the six processes may be interpreted physically in various ways. Variation of G will be taken to correspond to duct area change, although it could equally well correspond to change in the mass flow due to injection or withdrawal of fluid. Typical interpretations of the six processes appear in table 1.

Constant	Typical interpretation
H, s	Reversible, adiabatic ' nozzle ' process with area change.
F, G	Rayleigh process. Frictionless, constant area process with heat exchange or Ohmic heating, etc.
G, H	Fanno process. Adiabatic, constant area process with force in direction of motion due to wall friction, wire mesh, electromagnetic forces in ionized gas, etc. No external energy exchange.
G, s	Reversible, adiabatic work exchange in a duct of constant area by actuator discs, electromagnetic induction, etc.
F, s	Reversible, adiabatic process where work exchange and area change combine to keep F constant.
F, H	Adiabatic ' nozzle ' process in which F is constant owing to friction.

Table 1.

The last two cases are rather artificial, and are more interesting mathematically than practically. There is difficulty in interpreting F when G is not constant. More artificial interpretations of the first four processes are also possible. For example, the Fanno process can represent a reversible process with suitably synchronized work and heat exchanges.

There being only three independent properties of a flow-section, F, G, H and s must be related. The differential equation expressing this relation is

$$dF+\rho T\,ds = \rho\,dH+u\,dG, \tag{1}$$

which follows from the definition of the variables and from the equilibrium relation $T\,ds = dh-dp/\rho$, T being the absolute temperature. Denoting the invariance of quantities by suffices, we then have

$$dF_{Hs} = u\,dG_{Hs}, \quad \text{etc.} \tag{2}$$

There are six such relations each involving two differentials. We shall proceed to relate all twelve differentials and show in particular that all vanish when the state is sonic. In a process with two of F, G, H and s constant, at points where the remaining two variables are stationary, the small change occurring between two closely adjacent flow-sections having the same values of F, G, H and s is virtually a stationary sound wave. Thus M, the local Mach number, must be unity. M, or u/a, where a is the isentropic sound speed $\sqrt{(\partial p/\partial \rho)_s}$, is another gas dynamical property of each flow-section. A flow-section in a given state can be imagined as lying within any of innumerable processes, including the six currently under consideration.

From (1), $u = (\partial F/\partial G)_{Hs}$ and $\rho = (\partial F/\partial H)_{Gs}$, and hence

$$(\partial u/\partial H)_{Gs} = (\partial \rho/\partial G)_{Hs}. \tag{3}$$

There is a large array of these Maxwell-type relations, from which it is evident that all the differentials dH_{Gs}, dG_{Hs}, etc., will vanish simultaneously for flow-sections in certain states, provided there is no irregular behaviour of the quantities u, ρ, etc. Moreover, it is easily shown that

$$dG_{Hs} = \rho(1-M^2)\, du, \tag{4}$$

so that sonic states are the relevant ones as expected.

To provide a firmer basis for the analysis, it is convenient to take u as the major independent variable. Quantities of the type $(\partial G/\partial u)_{Hs}$, written G'_{Hs}, will be studied and related.

From (3)

$$H'_{Gs} = G'_{Hs}\left(\frac{\partial u}{\partial \rho}\right)_{Hs} = -\frac{G'_{Hs}}{u}\left(\frac{\partial h}{\partial \rho}\right)_s = -\frac{uG'_{Hs}}{\rho M^2}. \tag{5}$$

All the differential coefficients may be related in this way or by shorter alternative methods.

It is easily shown with the help of (2) that

$$\frac{F'_{GH}}{F'_{Gs}} = \frac{(\partial H/\partial s)_{FG}}{(\partial H/\partial s)_{Gu}} = \frac{T}{(\partial h/\partial s)_\rho}, \tag{6}$$

and that

$$\frac{H'_{FG}}{H'_{Gs}} = \frac{(\partial F/\partial s)_{GH}}{(\partial F/\partial s)_{Gu}} = -\frac{\rho T}{(\partial p/\partial s)_\rho}. \tag{7}$$

Also,

$$G'_{Fs} = G'_{Hs} + \left(\frac{\partial G}{\partial H}\right)_{su} H'_{Fs} = G'_{Hs} + \left(\frac{\rho u}{a^2}\right)\left(-\frac{uG'_{Fs}}{\rho}\right).$$

Hence

$$G'_{Fs}(1+M^2) = G'_{Hs}. \tag{8}$$

Similarly,

$$G'_{FH} = G'_{Hs} + \left(\frac{\partial G}{\partial s}\right)_{Hu} s'_{FH} = G'_{Hs} + \left\{-\rho T u\left(\frac{\partial \rho}{\partial p}\right)_h\right\}\left(\frac{uG'_{FH}}{\rho T}\right).$$

Hence

$$G'_{FH}\left\{1+u^2\left(\frac{\partial \rho}{\partial p}\right)_h\right\} = G'_{Hs}. \tag{9}$$

Equations (2) and (4) to (9), now permit all the derivatives with respect to u to be evaluated. Table 2 lists them as multiples of $(1-M^2)$. The example $H'_{Gs} = -u(1-M^2)/M^2$ illustrates the use of the table.

It is necessary to study the behaviour of the quantities appearing in the table before the criticality of the sonic state can be clearly asserted. M^2 is well-behaved, as the speed of sound must always be defined in real substances. The quantity appearing in the (F, G) Rayleigh case is

$$T\left(\frac{\partial s}{\partial p}\right)_\rho = -\frac{c_p}{a^2}\left(\frac{\partial T}{\partial \rho}\right)_p, \tag{10}$$

which is well-behaved except at such rare points as water at 4° C at atmospheric pressure, where $(\partial T/\partial \rho)_p$ tends to infinity. C_p, the specific heat at constant pressure, must be positive for stability. This is consistent with the findings of Kline & Shapiro (1954) who pointed out the criticality of the sign of $(\partial p/\partial s)_v$ and hence of $(\partial s/\partial p)_\rho$. The specific volume v is the reciprocal of ρ. The quantity appearing in the (F, H)-case is

Constant	G'/ρ	$F'/\rho u$	Ts'/u	H'/u
H, s (isentropic 'nozzle')	1	1	0	0
G, H (Fanno)	0	$-\frac{T}{M^2}\left(\frac{\partial s}{\partial h}\right)_\rho$	$\frac{T}{M^2}\left(\frac{\partial s}{\partial h}\right)_\rho$	0
F, G (Rayleigh)	0	0	$\frac{\rho T}{M^2}\left(\frac{\partial s}{\partial p}\right)_\rho$	$\frac{\rho T}{M^2}\left(\frac{\partial s}{\partial p}\right)_\rho$
G, s	0	$-M^{-2}$	0	$-M^{-2}$
F, s	$(1+M^2)^{-1}$	0	0	$-(1+M^2)^{-1}$
F, H	$\frac{1}{1+u^2(\partial\rho/\partial p)_h}$	0	$\frac{1}{1+u^2(\partial\rho/\partial p)_h}$	0

Table 2. Derivatives with respect to u as multiples of $(1-M^2)$.

$$\left(\frac{\partial \rho}{\partial p}\right)_h = \frac{1}{a^2} - \frac{1}{\rho c_p}\left(\frac{\partial \rho}{\partial T}\right)_p, \tag{11}$$

of which both terms are positive, except in states where there is thermal contraction. Apart from this possibility, $(\partial \rho/\partial p)_h$ is finite, positive and non-zero. The quantity appearing in the (G, H) Fanno case,

$$T\left(\frac{\partial s}{\partial h}\right)_\rho = \frac{1}{a^2}\left(\frac{\partial p}{\partial \rho}\right)_h, \tag{12}$$

is seen to be finite and positive except conceivably in a thermal contraction region, by reference to (11) above. We may note that

$$\left(\frac{\partial h}{\partial s}\right)_\rho = T + v\left(\frac{\partial p}{\partial s}\right)_v,$$

which is the quantity studied by Kline & Shapiro (1954) in their discussion of Fanno processes. They considered it very unlikely that $T+v(\partial p/\partial s)_v$ would ever be negative for real fluids.

Summing up, we may say that if the above special states are excluded, the pair of the variables F, G, H and s that are not kept constant are stationary when and only when $M=1$.

The next question to consider is whether these stationary values are maxima or minima. Kline & Shapiro (1954) have studied the (H,s), (F,G) and (G,H) cases and have shown separately in each case that the sign of $(\partial^2 p/\partial v^2)_s$ is the crucial factor. This also decides whether compressive acoustic waves steepen into shocks, and whether shocks must be compressive.

The six cases, keeping pairs of F, G, H and s constant, will now be treated in parallel to determine the nature of the stationary values as u varies. To fix the change of sign of F', G', H' or s' at the sonic state, it is necessary to study $\partial M^2/\partial u$ at the sonic state. It is found that $\partial M^2/\partial u$ takes the same value in all six cases when $M=1$. This follows from the five equations:

$$\left.\begin{aligned}\left(\frac{\partial M^2}{\partial u}\right)_{GX} &= \left(\frac{\partial M^2}{\partial u}\right)_{Gs} + \left(\frac{\partial M^2}{\partial s}\right)_{Gu} s'_{GX},\\ \left(\frac{\partial M^2}{\partial u}\right)_{sX} &= \left(\frac{\partial M^2}{\partial u}\right)_{Gs} + \left(\frac{\partial M^2}{\partial G}\right)_{su} G'_{sX},\end{aligned}\right\} X = F \text{ or } H$$

and
$$\left(\frac{\partial M^2}{\partial u}\right)_{FH} = \left(\frac{\partial M^2}{\partial u}\right)_{Hs} + \left(\frac{\partial M^2}{\partial s}\right)_{Hu} s'_{FH},$$

in which the last terms vanish when $M=1$, provided $(\partial M^2/\partial s)_{Gu}$, $(\partial M^2/\partial G)_{su}$, $(\partial M^2/\partial s)_{Hu}$ are not infinite when $M=1$. These quantities are respectively equal to

$$-\frac{1}{a^2}\left(\frac{\partial a^2}{\partial s}\right)_\rho, \qquad -\frac{1}{a^3}\left(\frac{\partial a^2}{\partial \rho}\right)_s, \qquad -\frac{1}{a^2}\left(\frac{\partial a^2}{\partial s}\right)_h$$

when $M=1$. Now

$$\left(\frac{\partial a^2}{\partial s}\right)_\rho = \frac{\partial^2 p}{\partial \rho\, \partial s} = v^2\left(\frac{\partial^2 T}{\partial v^2}\right)_s \quad \text{and} \quad \left(\frac{\partial a^2}{\partial \rho}\right)_s = \left(\frac{\partial^2 p}{\partial \rho^2}\right)_s,$$

and no singularity upon isentropics, such as would make these second derivatives infinite, appears to be possible. Furthermore,

$$\left(\frac{\partial a^2}{\partial s}\right)_h = \left(\frac{\partial a^2}{\partial s}\right)_\rho + \left(\frac{\partial a^2}{\partial \rho}\right)_s \left(\frac{\partial \rho}{\partial s}\right)_h \quad \text{and} \quad \left(\frac{\partial \rho}{\partial s}\right)_h = -\rho T\left(\frac{\partial \rho}{\partial p}\right)_h.$$

Equation (11) indicates that $(\partial\rho/\partial p)_h$ is not infinite, and so $(\partial a^2/\partial s)_h$ is not infinite.

We conclude that $\partial M^2/\partial u$ takes the same value when $M=1$, whichever pair of F, G, H and s is kept constant. The common value is most easily evaluated in the (G,s) case, where u/v is constant:

$$\left(\frac{\partial M^2}{\partial u}\right)_{Gs} = \frac{v}{u}\frac{\partial}{\partial v}\left\{-\frac{u^2}{v^2}\frac{1}{(\partial p/\partial v)_s}\right\}_s = \frac{u}{v}\frac{1}{(\partial p/\partial v)_s^2}\left(\frac{\partial^2 p}{\partial v^2}\right)_s = \frac{v^3}{a^3}\left(\frac{\partial^2 p}{\partial v^2}\right)_s$$

when $M=1$.

Kline & Shapiro (1954) indicated that $(\partial^2 p/\partial v^2)_s$ is positive for all single phase fluids for which accurate data are available. We therefore assume that $(\partial^2 p/\partial v^2)_s$ is positive, and also that there is no thermal contraction, in what follows. It is then possible, using (10), (11) and (12), to deduce the nature of the stationary values of F, G, H and s at sonic points from the change in sign in the relevant derivative with respect to u. Table 3 lists the results. It should be observed that the sonic points have the same nature as for a perfect gas.

Constant	Nature of sonic points
H, s (isentropic nozzle)	F max.; G max.
G, H (Fanno)	F min.; s max.
F, G (Rayleigh)	H max.; s max.
G, s	F min.; H min.
F, s	G max.; H min.
F, H	G max.; s max.

Table 3.

If we assume that the variation of M, F, G, H and s with u in the above cases is continuous, it follows that only one sonic point occurs in each process (if $(\partial^2 p/\partial v^2)_s$ is fixed in sign) since maxima and minima must occur alternately. (Sonic points of inflexion would require $\partial M^2/\partial u$ and $(\partial^2 p/\partial v^2)_s$ to vanish.) This fact was deduced by Kline & Shapiro (1954) from a different argument in the Fanno and Rayleigh cases.

3. THE INFLUENCE OF THE SPEED OF NEWTONIAN SOUND

Newton's calculation of the speed of sound in air proved to be too low because of his assumption that sound propagation was an isothermal process instead of an isentropic one. Nevertheless the speed of Newtonian sound, which will be denoted by n to distinguish it from the adiabatic sound speed a, proves to have some physical significance.

The symbol N will be used for the 'isothermal Mach number', the ratio u/n. The specific heat ratio γ is equal to N^2/M^2 because

$$\frac{a^2}{n^2} = \frac{(\partial p/\partial \rho)_s}{(\partial p/\partial \rho)_T} = \frac{(\partial s/\partial T)_p}{(\partial s/\partial T)_\rho} = \frac{c_p}{c_v} = \gamma.$$

γ is always greater than unity since $c_p - c_v = -T(\partial v/\partial T)_p^2(\partial p/\partial v)_T$ and $(\partial p/\partial v)_T$ must be negative for stability. γ reaches unity only at points of zero thermal expansion where $(\partial v/\partial T)_p$ vanishes. In passing, it it worth noting that the speeds of adiabatic and Newtonian sound are $(\gamma p/\rho)^{1/2}$ and $(p/\rho)^{1/2}$ for any fluid which obeys Boyle's law, $p/\rho = f(T)$. A reactive gas mixture whose mean molecular weight changes does not satisfy this condition.

It is well known that, for a perfect gas undergoing a Rayleigh process, the temperature has a stationary value when $M = \gamma^{-1/2}$ and $N = 1$. This, however, is a general result, applicable to all single phase fluids in equilibrium, for

$$u^2 = \left(\frac{\partial p}{\partial \rho}\right)_{FG} \quad \text{and } a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s.$$

It follows that

$$\left(\frac{\partial T}{\partial p}\right)_s \left(\frac{\partial p}{\partial s}\right)_{FG} = M^2 \left(\frac{\partial T}{\partial \rho}\right)_s \left(\frac{\partial \rho}{\partial s}\right)_{FG}$$

and hence that

$$\left(\frac{\partial T}{\partial s}\right)_{FG} - \left(\frac{\partial T}{\partial s}\right)_p = M^2 \left\{\left(\frac{\partial T}{\partial s}\right)_{FG} - \left(\frac{\partial T}{\partial s}\right)_\rho\right\}.$$

From this we have

$$\left(\frac{\partial T}{\partial s}\right)_{FG} = \frac{T}{c_p} \frac{1-\gamma M^2}{1-M^2},$$

which vanishes when $M = \gamma^{-1/2}$. If the fluid is a reactive gas mixture, γ must be taken as the *effective* specific heat ratio, allowing for reaction.

A flow-section such that $N = 1$, i.e. capable of sustaining a weak steady Newtonian sound wave in which F, G and T do not change significantly, obviously corresponds to a state where T is stationary if F and G are fixed as in a Rayleigh process. It also corresponds to the section when G and the duct area have stationary values in a reversible, isothermal nozzle process. In fact there are again six simple processes for which n is the critical speed. A unified, symmetrical treatment is again possible if a new property Z, a dynamic form of Gibbs function, is introduced such that

$$Z = H - Ts = \zeta + \tfrac{1}{2}u^2,$$

ζ being the usual Gibbs function per unit mass, $h - Ts$. Z is not the stagnation Gibbs function, which is $H - T_0 s$, where T_0 is the stagnation temperature. Z has no obvious physical significance except in isothermal cases where its change measures reversible work exchanges (see §4). Thus Z is constant in the reversible, isothermal nozzle process. The lack of significance of Z is related to the fact that the datum for s is arbitrary and therefore Z contains an arbitrary multiple of T when T varies. The two cases where Z, but not T, is constant will not be considered further.

Constant	Typical interpretation
F, G	Rayleigh process, as already discussed.
T, Z	Isothermal, reversible ‘ nozzle ’ process with area change.
G, T	Isothermal, constant area process with state change produced by wall friction or work exchange.
F, T	Isothermal process in which area change and wall friction or work exchange combine to keep F constant.

Table 4.

The four variables F, G, T and Z are related by the equation

$$dF = \rho\, dZ + \rho s\, dT + u\, dG. \tag{13}$$

The four significant cases in which pairs of the variables are kept constant are interpreted in table 4.

This situation may be analysed in a manner completely analogous to that in the foregoing section. For brevity, only the main results will be presented. Again u is taken as the main independent variable and primes denote differentiation with respect to u. Table 5 lists these derivatives as multiples of $(1-N^2)$.

Constant	G'/ρ	$F'/\rho u$	sT'/u	Z'/u
T, Z (isothermal 'nozzle')	1	1	0	0
F, G (Rayleigh)	0	0	$\frac{\rho s}{N^2}\left(\frac{\partial T}{\partial p}\right)_\rho$	$-\frac{\rho s}{N^2}\left(\frac{\partial T}{\partial p}\right)_\rho$
G, T	0	$-N^{-2}$	0	$-N^{-2}$
F, T	$(1+N^2)^{-1}$	0	0	$-(1+N^2)^{-1}$

Table 5. Derivatives with respect to u as multiples of $(1-N^2)$.

It is necessary to study the behaviour of the quantities appearing in table 5 before the criticality of the Newtonian sonic state can be clearly asserted. N^2 is well behaved, since two-phase states in which $(\partial p/\partial\rho)_T$ and n^2 vanish are excluded. Also,

$$(\partial T/\partial p)_\rho/N^2 = -(\partial T/\partial\rho)_p/u^2,$$

which is positive and finite if thermal contraction is excluded. Note that the factor s also occurs in the quantity sT'/u so that its sign is irrelevant to the behaviour of T in the Rayleigh case.

The conclusion is that with the above restrictions the pair of the variables F, G, T and Z that are not kept constant are stationary when and only when $N=1$. The nature of these stationary points depends upon the sign of $\partial N^2/\partial u$ there.

In the (G, T) case, it is easily shown that

$$\frac{\partial N^2}{\partial u} = \frac{v^3}{n^3}\left(\frac{\partial^2 p}{\partial v^2}\right)_T \quad \text{when } N=1. \tag{14}$$

It is also easily demonstrated that $\partial N^2/\partial u$ takes this same value at the Newtonian sonic point in all four cases. The analysis is similar to that in §2, with N, n, T, Z and ζ replacing M, a, s, H and h. The condition for $\partial N^2/\partial u$ to be identical is found to be that $(\partial n^2/\partial T)_\rho$, which is equal to $-v^2(\partial^2 s/\partial v^2)_T$, and $(\partial n^2/\partial\rho)_T$, which is equal to $(\partial^2 p/\partial\rho^2)_T$, should not be infinite. No singularity upon isothermals such as would make these quantities infinite appears to be possible. We conclude that in all four cases (14) holds. The derivative $(\partial^2 p/\partial v^2)_T$ is usually positive, but can become negative near the critical point for pure fluids. If we assume that the positive sign prevails, and that there is no thermal contraction, we can ascertain the nature of the stationary values when $N = 1$. The results appear in table 6. The behaviour is qualitatively like that of a perfect gas for substances for which there is no thermal contraction and for which $(\partial^2 p/\partial v^2)_T$ is positive.

Constant	Nature of Newtonian sonic points
F, G (Rayleigh)	T max.; Z min. (if $s > 0$)
T, Z (isothermal nozzle)	F max.; G max.
G, T	F min.; Z min.
F, T	G max.; Z max.

Table 6.

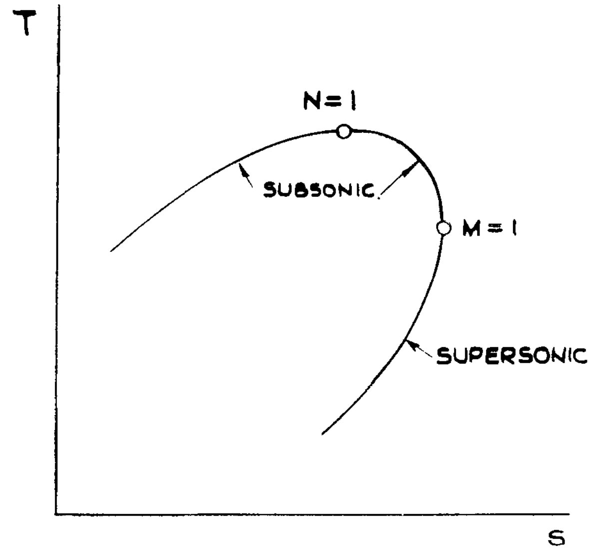

Figure 1. The Rayleigh Line, for the case $(\partial^2 p/\partial v^2)_{s \text{ or } T} > 0$, $(\partial \rho/\partial T)_p < 0$.

If we assume that the variation of F, G, T and Z with u in the above cases is continuous, it follows that only one point at which $N = 1$ occurs in each process (if $(\partial^2 p/\partial v^2)_T$ is fixed in sign) since maxima and minima must occur alternately and points of inflexion would require $(\partial^2 p/\partial v^2)_T$ to vanish.

This, and the corresponding result for points at which $M = 1$, constrains a Rayleigh line on the (T, s)-plane always to have the form familiar for the perfect gas case, illustrated in figure 1. The subsonic and supersonic branches cannot cross because this would imply two flow-sections with the same values of T and s, and hence in the same thermodynamic state with the same values of a and p, but with differing Mach numbers and velocities. But $F = p + Gu$, F and G being constant, which shows that for a given value of p, u can have only one value. Thus a crossing is precluded.

4. SOME OTHER CONSIDERATIONS

When propulsive effort is the prime concern, the quantity

$$(p + \rho u^2) \times (\text{duct area of cross-section})$$

is of interest. This quantity is represented by F/G per unit rate of mass flow, $1/G$ being the area per unit rate of mass flow. It is possible to develop the analysis using the four variables F/G, $1/G$, H and s, which are related by the equation

$$ud(F/G) + T\,ds = dH + pu\;d(1/G).$$

The results differ little from those in § 2, there being only two fresh processes involved. These are the cases where F/G and H or s are constant. For these cases $1/G$ and s or H, respectively, are stationary at the sonic point.

Another approach is offered by the selection of the four variables H, p, u and s for analogous treatment. The basic relation here is

$$\rho\,dH = dp + \rho T\,ds + G\,du.$$

No critical point like the sonic point occurs in this case, however. For instance, when H and s are fixed, p is stationary only when u and G vanish. Another case belonging to this family is that where s, p and hence the thermodynamic state are fixed. It corresponds to cases where the kinetic energy is converted reversibly into work by idealized impulse turbines or electromagnetic induction. Again the sonic point is not critical.

There are obviously many other similar, alternative selections of variables possible.

Interpretating the individual cases physically reveals an asymmetry in the way that heat and work influence the problem. In reversible cases, the external heat exchanged between two adjacent flow-sections is $T\,ds$ per unit mass flow, but the corresponding work quantity cannot be expressed in terms of any single perfect differential in general. Alternative, general expressions for dW, the external work exchanged between two adjacent flow-sections per unit mass flow in a reversible process, are

$$dW = T\,ds - dH = -(u\,du + dp/\rho) = (u\,dG - dF)/\rho = -dZ - s\,dT.$$

The difference between dW and the actual work exchanged in an irreversible case is the dissipation. In reversible cases without work exchange such as the two nozzle processes and the Rayleigh process, dW vanishes. This will be recognized as indicating that Euler's equation

$$u\,du + dp/\rho = 0$$

is applicable.

Gas dynamical analysis frequently makes use of the stagnation state corresponding to a given flow-section. This state, usually denoted by the suffix 0, is defined as that for which

$$u_0 = 0, \qquad h_0 = H, \qquad s_0 = s.$$

The stagnation temperature T_0 is of interest mainly for substances for which h is a function of temperature only, as for a perfect gas. For all normal fluids T_0 is larger than T, the 'static' temperature, since H is always greater than h, and

$$\left(\frac{\partial T}{\partial h}\right)_s = -\frac{T}{\rho c_p}\left(\frac{\partial \rho}{\partial T}\right)_p > 0$$

provided thermal contraction is excluded.

Variation of the stagnation pressure p_0 is given by the relation

$$dp_0 = \rho_0(dH - T_0\, ds).$$

p_0 is obviously constant in the isentropic nozzle process. In a Fanno process, H is constant and

$$dp_0 = -\rho_0 T_0\, ds.$$

Thus if the process is irreversible and adiabatic, p_0 must decrease.

In a Rayleigh process, $dH = T\,ds$ and

$$dp_0 = -\rho_0(T_0 - T)\, ds. \tag{15}$$

Thus the sonic point is the point of minimum p_0 for normal fluids without thermal contraction and for which $(\partial^2 p/\partial v^2)_s$ is positive.

In the process with s and F or G constant, p_0 behaves qualitatively like H, reaching a minimum at the sonic point for normal fluids. In the (F, H) process it is again minimum when $M = 1$ since s is then a maximum.

Thus, of the six F, G, H, s cases, p_0 is constant in one case and is minimal at the sonic point in all the others for normal fluids.

In the isothermal nozzle process, $dH = T\,ds$ and (15) again applies. Neither p_0 nor s has stationary points in this case, however.

Across a stationary shock, H is constant whereas s always increases. It follows that p_0 always decreases because $(\partial p_0/\partial s)_H = -\rho_0 T_0 < 0$.

5. CONCLUDING REMARKS

One conclusion of the analysis is to confirm and generalize the result of Kline & Shapiro (1954) that the gas-dynamical behaviour of single-phase fluids in equilibrium is qualitatively similar to that of a perfect gas, provided that there is no thermal contraction and that $(\partial^2 p/\partial v^2)_s$ is positive. The additional demand that $(\partial^2 p/\partial v^2)_T$ be positive is made.

The speeds of adiabatic and Newtonian sound are seen to be significant since a flow-section where the velocity is sonic is capable of undergoing an infinitesimal acoustic-type change in which F and G together with either s and H (if $M = 1$) or T and Z (if $N = 1$) do not change. It is then apparent, for instance, why in a Rayleigh process with F and G constant, s or T is stationary at the two types of sonic point.

The tendency of nozzle, Rayleigh and Fanno processes to choke when the velocity becomes sonic is well known. The similar choking occurring in the (G, s), reversible, adiabatic work-exchange process is perhaps a new result. The minimum value of H which occurs at the sonic point means that a limit is placed on the work that may be extracted reversibly in a duct of constant cross-sectional area. The comparable process mentioned in § 4, in which the duct area varies in such a way that work extraction changes only the kinetic energy and not the thermodynamical state of the fluid, shows no choking tendency. The limit on the work extraction here is provided by the growth in duct area as the velocity falls.

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KLINE, S. J. & SHAPIRO, A. H. 1954 Article in *Mémoires sur la Méchanique des Fluides*, p. 171. Publ. Sci. Tech. Minist. de l'Air, Paris.

SHAPIRO, A. H., & HAWTHORNE, W. R. 1947 *J. Appl. Mech.* **14**, 317.

CORRIGENDUM

"Wave formation in laminar flow down an inclined plane", by T. BROOKE BENJAMIN (*J. Fluid Mech.* **2**, 1957, 554).

Page 560. In equation (3.11) the equals sign following the first term should be replaced by a minus sign.

Page 570. In equation (5.10) the numerical factor should be replaced by 0·448, and in equation (5.12) the numerical factor should be replaced by 0·00783. Equation (5.13) should read as

$$\mathscr{A} = \exp\{0{\cdot}108R^{7/3}\}.$$